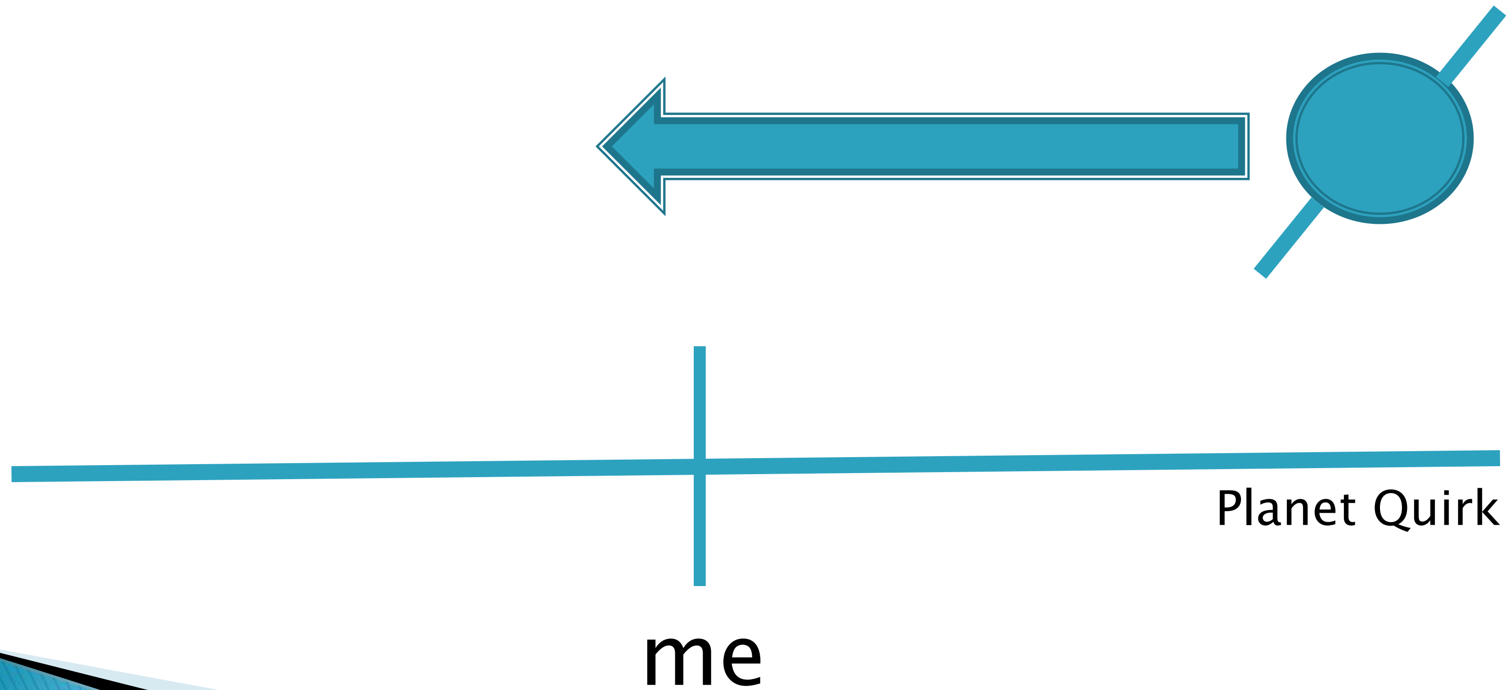
Application of Linear Approximation

Scott Starks, PhD, PE
Professor of Electrical & Computer Engineering
UTEP

(Based upon material published in MIT Open Course Ware
http://ocw.mit.edu/courses/mathematics/18-01sc-single-variable-calculus-fall-2010/unit-2-applications-of-differentiation/part-a-approximation-and-curve-sketching/session-24-examples-of-linear-approximation/MIT18_01SCF10_Ses24d.pdf

Time Dilation

- Let's imagine that I am on Planet Quirk.
- A satellite is whizzing overhead with a velocity v.



Timing Information

- The satellite has a clock on it that reports a time, T.
- I have a watch that reports a time, T_m.

Special Relativity

- The following equation from special relativity gives the relationship between the two times:

$$T_m = \frac{T}{\sqrt{1-\frac{v^2}{c^2}}}$$

- I would like to know how different are the two times?

Substitution of Variables

- Let's make a substitution of variables

$$u = \frac{v^2}{c^2}$$

$$T_m = \frac{T}{\sqrt{1-u}} = T\left(\frac{1}{\sqrt{1-u}}\right) = T \times f(u)$$

- where

$$f(u) = \frac{1}{\sqrt{1-u}}$$

Find the Derivative

$$f'(u) = \frac{d\left(\frac{1}{\sqrt{1-u}}\right)}{du} = \left(-\frac{1}{2}\right)(1-u)^{-\frac{3}{2}}(-1)$$

$$= \boxed{\frac{1}{2}(1-u)^{-\frac{3}{2}}}$$

Linear Approximation

- Formula for linear approximation

$$f(u) = \frac{1}{\sqrt{1-u}} \approx f(a) + f'(a)(u-a)$$

- We will take the approximation about the value $a = 0$

$$f(u) \approx f(a) + f'(a)(u-a)\Big|_{a=0}$$

$$\approx f(0) + f'(0)u$$

$$f(0) = 1$$

$$f'(0) = \frac{1}{2}(1-u)^{-\frac{3}{2}}\Bigg|_{u=0} = \frac{1}{2}$$

$$f(u) \approx 1 + \frac{1}{2}u$$

Linear Approximation for T_m

$$T_m = \frac{T}{\sqrt{1-u}} \approx T\left(1 + \frac{1}{2}u\right)$$

$$\approx T\left(1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right)$$

Why is this relevant?

- GPS transmitters are mounted on satellites that are in orbit above the Earth.
- GPS receivers are omni-present on Earth. Many of these are in motion (autos) also.
- According to special relativity, there will be a difference between the time on your GPS receiving device and the time on the GPS satellite.
- This difference will impact your receiving device's estimate of position.

- The engineers who set up the GPS satellite system anticipated this difference in time.
- GPS satellites move at about 4 km/s.
- The speed of light is 3 X 10^5 km/s and T_m is approximately T X (1.0000000005).
- There is hardly any difference between the times measured on the ground and on the satellite.

- Because v^2/c^2 is very close to 0, our linear approximation should be very close to the actual value of T_m.
- Another good reason for using linear approximation here is that if the answer is "the difference is too small to matter", the person doing the calculation has no use for a more precise answer which may be more difficult to calculate.

- Nonetheless, engineers used this very approximation (along with several other approximations) to calibrate the radio transmitters on GPS satellites.